

Two-Component Spinors in Spacetimes with Torsionful Affinities

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Abstract

The essentially unique torsionful version of the classical two-component spinor formalisms of Infeld and van der Waerden is presented. All the metric spinors and connecting objects that arise here are formally the same as the ones borne by the traditional formalisms. Any spin-affine connexion appears to possess a torsional part which is conveniently chosen as a suitable asymmetric contribution. Such a torsional affine contribution thus supplies a gauge-invariant potential that can eventually be taken to carry an observable character, and thereby effectively takes over the role of any trivially realizable symmetric contribution. The overall curvature spinors for any spin-affine connexion accordingly emerge from the irreducible decomposition of a mixed world-spin object which in turn comes out of the action on elementary spinors of a typical torsionful second-order covariant derivative operator. Explicit curvature expansions are likewise exhibited which fill in the gap related to their absence from the literature. It is then pointed out that the utilization of the torsionful spinor framework may afford locally some new physical descriptions.

1 Introduction

A remarkable property of Einstein-Cartan's gravitational theory [1-4] relies upon the fact that the characteristic asymmetry of the Ricci tensor for any torsionful world affine connexion always entails the presence of asymmetric sources on the right-hand sides of the field equations. These sources were traditionally identified [5, 6] with local densities of intrinsic angular momentum of matter.

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Since the advent of Einstein-Cartan's theory, several attempts have been made at designing torsional versions of extended approaches to gravity that might circumvent the issues related to some cosmological problems while supplying a macroscopic explanation of the presently observable acceleration of the universe (see, for instance, Refs. [7-10]). As brought forward by Refs. [11-15], torsional gravity has by itself attracted much attention from researchers in conjunction with a prediction accomplished by string theory that concerns the occurrence of couplings between torsion and spinning fields. Amongst the developments that have arisen from this field theoretical situation, noticeably enough, is the work of Ref. [12] which provides a scheme that helps understand the ratios between the coupling strengths for all the fundamental interactions. It is thus shown that the value of a typical torsion-coupling constant can pass through those of the coupling constants for the other interactions during the cosmic evolution. Another noteworthy work posed in this connection [13], allows for a family of leptons within a torsional gravitational framework and establishes that the torsionic property of the underlying spacetime geometry may give rise to interactions having the structure of the weak forces. Moreover, for the case of a non-linear Lagrangian density for the gravitational sector, the corresponding coupling configurations appear to generate both the structure and the strength of the electroweak interactions among leptons. It turned out, then, that the weak interactions among the leptons effectively taken into consideration could be regarded as a geometric effect due to couplings between torsion and spinor fields.

The two-component spinor framework for classical general relativity is constituted by the so-called $\gamma\varepsilon$ -formalisms of Infeld and van der Waerden [16]. This framework was primarily aimed at describing the dynamics of classical Dirac fields in curved spacetimes, with its construction having been carried out much earlier than the achievement of the definitive conditions for a curved space to admit spinor structures locally [4]. Thus, the basic procedure just involves setting up two pairs of conjugate spin spaces at every non-singular point of a curved spacetime that is endowed with a torsionless covariant derivative operator. Furthermore, the generalized Weyl gauge group [17] operates locally on any spin spaces in a way that does not depend at all upon the action of the pertinent manifold mapping group. One of the key assumptions lying behind the original construction of the formalisms amounts to taking any Hermitian connecting objects as covariantly constant entities. The implementation of this assumption readily produces in either formalism a self-consistent set of world-spin metric and affine correlations [18]. All the corresponding curvature spinors arise most simply from the decomposition of mixed world-spin quantities that result out of the action of covariant derivative commutators on arbitrary spin vectors [19]. Loosely speaking, the most striking physical feature of any such curvature spinors lies over the fact that they are given as sums of purely gravitational and electromagnetic contributions which bring forth in an inextricably geometric fashion the occurrence of wave functions for gravitons and photons of both handednesses. A fairly complete version of the $\gamma\varepsilon$ -framework is given in Ref. [18]. The gravitational contributions for the ε -formalism were utilized in

Ref. [20] to support a spinor translation of Einstein's equations, but it had been established somewhat earlier [21] that any of them should show up as a spinor pair which must be associated to the irreducible decomposition of a Riemann tensor. In both the formalisms, any gravitational wave functions turn out to be ultimately defined as totally symmetric curvature pieces that occur in spinor decompositions of Weyl tensors [20]. Any electromagnetic curvature contribution, on the other hand, amounts to a pair of suitably contracted pieces that enter the spinor representation of a locally defined Maxwell bivector and satisfy a peculiar conjugation property [18, 19]. The propagation of gravitons for the ε -formalism, and the description of their couplings to external electromagnetic fields, were given in Refs. [4, 20]. Nevertheless, only recently [22, 23] has the full $\gamma\varepsilon$ -description of the propagation of spin curvatures in spacetime been obtained. It thus appears that the couplings between gravitons and photons are exclusively borne by the wave equations that govern the electromagnetic propagation. In Ref. [24], it was likewise suggested that a description of some of the physical properties of the cosmic microwave background may be achieved by looking at the propagation in Friedmann-like conformally flat spacetimes of Infeld-van der Waerden photons. The $\gamma\varepsilon$ -framework was extensively employed over the years by many authors in a more pragmatic way particularly to reconstruct some classical generally relativistic structures and to transcribe classification schemes for world curvature tensors [25-29]. However, the torsionful version of the formalisms has been sparsely considered in the literature just to a minor extent [30, 31].

The present work exhibits systematically the natural torsional extension of the classical $\gamma\varepsilon$ -formalisms. One of the motivations for elaborating our work is that it may certainly be of relevance for the framework of modified gravity theories. We will assume at the outset that the shift of any classical geometric considerations to the torsional context must preserve both the structure of manifold mapping groups and the form of the matrices that classically make out the Weyl gauge group. Hence, all the defining prescriptions for the world and spin densities involved in the old formalisms may be applicable equally well herein. Remarkably, the entire set of algebraic configurations carrying the metric spinors and connecting objects for the torsional formalisms, has the same form and gauge characterizations as the one for the Infeld-van der Waerden formalisms. In other words, the whole spinor algebra of the old framework is passed on without any formal changes to the new framework. A typical spin-affine connexion for either of the new formalisms carries additively a torsional piece which is conveniently chosen as a suitable asymmetric contribution. In contradistinction to any trivially realizable symmetric spin affinities, such a torsional affine contribution thus supplies a gauge-invariant potential that may carry an observable character. Because of the supposedly legitimate additivity of spin affinities and the general pattern of world-affine splittings, the classical world-spin affine correlations we had referred to previously remain all formally valid within the torsional framework, and thence also so do the classical covariant differential expansions for world and spin densities as well as the system of Infeld-van der Waerden metric eigenvalue equations [16]. The curvature spinors

for some spin-affine connexion occur in the decomposition of a characteristic mixed world-spin object that accordingly comes from the action on elementary spinors of a geometrically appropriate torsionful second-order covariant derivative operator. Explicit curvature expansions are then obtained on the basis of the use of well-known symbolic valence-reduction devices [4]. Indeed, the symmetry specification of the individual constituents of such expansions were supplied in Ref. [31], but the overall curvatures for the torsional framework have not been given hitherto.

Unless otherwise tacitly stated, the term "formalism" and the plural version of it shall henceforward designate the new framework. The notation adopted in Ref. [18] will be taken for granted except that spacetime components will now be labelled by lower-case Greek letters. In particular, we denote as x^μ some local coordinates on a spacetime \mathfrak{M} equipped with a torsionful covariant derivative operator ∇_μ . The partial derivative operator for x^μ is denoted as ∂_μ . Any world-metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ on \mathfrak{M} bear the traditional generally relativistic symmetry together with the local signature $(+ - - -)$, whence each of them still possesses 10 real independent components. We require $g_{\mu\nu}$ to fulfill the metric compatibility condition

$$\nabla_\mu g_{\lambda\sigma} = 0.$$

Usually, \mathfrak{M} should admit a local spinor structure in much the same way as for the classical case of general relativity. Without any risk of confusion, we will make use of the same indexed symbol ∇_μ for expressing covariant derivatives in both formalisms. The elements of the Weyl gauge group are non-singular complex (2×2) -matrices whose components are defined as

$$\Lambda_A{}^B = \sqrt{\rho} \exp(i\theta) \delta_A{}^B,$$

where $\delta_A{}^B$ denotes the Kronecker symbol, ρ stands for a positive-definite differentiable real-valued function of x^μ and θ amounts to the gauge parameter of the group which is taken as an arbitrary differentiable real-valued function on \mathfrak{M} . For the determinant of $(\Lambda_A{}^B)$, we have the expression

$$\det(\Lambda_A{}^B) \doteq \Delta_\Lambda = \rho \exp(2i\theta).$$

It will be expedient to recall in Section 2 some facts concerning torsional world geometry. This will considerably facilitate setting out some of the spin properties of immediate interest to us. In spite of the fact that the metric spinors and connecting objects for both formalisms are formally the same as the ones for the traditional framework, we shall have to introduce them into Section 3 along with the torsional spin affinities and some world-spin affine configurations. There, the spin-metric algebraic structures and affine devices for computing covariant derivatives of spin densities are only slightly touched upon, but we will place emphasis on the description of the patterns for the torsional affine contributions and their behaviours under gauge transformations. The spin curvatures of the formalisms are shown in Section 4. We draw an outlook from our work in Section 5. A few additional conventions will be explained in due course.

2 Torsional World Geometry

The world affine connexion associated with ∇_μ is split out as

$$\Gamma_{\mu\nu\lambda} = \tilde{\Gamma}_{\mu\nu\lambda} + T_{\mu\nu\lambda}, \quad (1)$$

where $\tilde{\Gamma}_{\mu\nu\lambda} \doteq \Gamma_{(\mu\nu)\lambda}$ may occasionally be identified with a Christoffel connexion, and $T_{\mu\nu\lambda} \doteq \Gamma_{[\mu\nu]\lambda}$ is the torsion tensor of ∇_μ . For some world-spin scalar f on \mathfrak{M} , we have the differential prescription

$$D_{\mu\nu}f = 0, \quad D_{\mu\nu} \doteq 2(\nabla_{[\mu}\nabla_{\nu]} + T_{\mu\nu}{}^\lambda\nabla_\lambda), \quad (2)$$

whence the operator $D_{\mu\nu}$ is linear and possesses the Leibniz rule property. It is obvious that $\tilde{\Gamma}_{\mu\nu\lambda}$ carries 40 real independent components whereas $T_{\mu\nu\lambda}$ carries 24. The covariant derivatives of some purely world vectors v^α and u_β are written down as

$$\nabla_\mu v^\lambda = \tilde{\nabla}_\mu v^\lambda + T_{\mu\sigma}{}^\lambda v^\sigma, \quad \nabla_\mu u_\lambda = \tilde{\nabla}_\mu u_\lambda - T_{\mu\lambda}{}^\sigma u_\sigma, \quad (3)$$

where $\tilde{\nabla}_\mu$ equals¹ the covariant derivative operator of $\tilde{\Gamma}_{\mu\nu\lambda}$. For v^λ , for example, we have

$$\tilde{\nabla}_\mu v^\lambda \doteq \partial_\mu v^\lambda + \tilde{\Gamma}_{\mu\sigma}{}^\lambda v^\sigma. \quad (4)$$

When acting on world-spin scalars, the operators ∇_μ and $\tilde{\nabla}_\mu$ must agree with each other in the sense that they should thus yield common results like $\partial_\mu f$. Consequently, the metric compatibility condition for $g_{\mu\nu}$ can be reexpressed as the expansion

$$\tilde{\nabla}_\lambda g_{\mu\nu} - 2T_{\lambda(\mu\nu)} = 0, \quad (5)$$

which is essentially equivalent to the relation

$$\Gamma_\mu = \tilde{\Gamma}_\mu + T_\mu = \partial_\mu \log(-\mathfrak{g})^{1/2}, \quad (6)$$

with $\Gamma_\mu \doteq \Gamma_{\mu\lambda}{}^\lambda$, for instance, and \mathfrak{g} standing for the determinant of $g_{\mu\nu}$.

The Riemann tensor for $\Gamma_{\mu\nu\lambda}$ occurs in either of the configurations

$$D_{\mu\nu}v^\lambda = R_{\mu\nu\sigma}{}^\lambda v^\sigma, \quad D_{\mu\nu}u_\lambda = -R_{\mu\nu\lambda}{}^\sigma u_\sigma, \quad (7)$$

and obeys the equality

$$R_{\mu\nu\lambda}{}^\rho = \tilde{R}_{\mu\nu\lambda}{}^\rho + R_{\mu\nu\lambda}^{(T)\rho} + 2(T_{[\mu|\tau|}{}^\rho \tilde{\Gamma}_{\nu]\lambda}{}^\tau + \tilde{\Gamma}_{[\mu|\tau|}{}^\rho T_{\nu]\lambda}{}^\tau), \quad (8)$$

where the expressions for $\tilde{R}_{\mu\nu\lambda}{}^\rho$ and $R_{\mu\nu\lambda}^{(T)\rho}$ may be obtained from the definition

$$R_{\mu\nu\lambda}{}^\rho \doteq 2(\partial_{[\mu}\Gamma_{\nu]\lambda}{}^\rho + \Gamma_{[\mu|\tau|}{}^\rho \Gamma_{\nu]\lambda}{}^\tau), \quad (9)$$

just by putting the kernel letters $\tilde{\Gamma}$ and T in place of Γ , respectively. We should stress, however, that $R_{\mu\nu\lambda}^{(T)\rho}$ does *not* constitute a tensor by itself, but the sum

¹The tensor $T_{\mu\lambda\sigma}$ of Ref. [4] conventionally equals (-2) times ours.

of it with the crossed $\tilde{\Gamma}T$ -terms of Eq. (8) does. We will consider further this world characterization later in Section 5.

It should be clear that $R_{\mu\nu\lambda\sigma}$ bears skewness in the indices of each of the pairs $\mu\nu$ and $\lambda\sigma$, but the Riemann-Christoffel index-pair symmetry does not take place here. Therefore, $R_{\mu\nu\lambda\sigma}$ possesses 36 real independent components while its Ricci tensor possesses 16. It can then be said that the Ricci tensor for any affine connexion of the type specified by Eq. (1), carries asymmetry. Some symbolic computations easily show that the role of the classical cyclic property of Riemann-Christoffel tensors has hereupon to be taken over by

$$R_{[\mu\nu\lambda]}{}^\sigma - 2\nabla_{[\mu}T_{\nu\lambda]}{}^\sigma + 4T_{[\mu\nu}{}^\tau T_{\lambda]\tau}{}^\sigma = 0, \quad (10)$$

whilst the Bianchi identity should now read

$$\nabla_{[\mu}R_{\nu\lambda]\sigma}{}^\rho - 2T_{[\mu\nu}{}^\tau R_{\lambda]\tau\sigma}{}^\rho = 0. \quad (11)$$

The property $R_{\mu\nu\lambda\sigma} = R_{[\mu\nu][\lambda\sigma]}$ and the torsionless relation $R_{[\mu\nu\lambda]\sigma} = 0$ entail imparting² the index-pair symmetry to $R_{\mu\nu\lambda\sigma}$. By invoking the dualization schemes given in Ref. [4], and making some index manipulations thereafter, we rewrite Eqs. (10) and (11) as

$${}^*R^\lambda{}_{\mu\nu\lambda} + 2\nabla^{\lambda*}T_{\lambda\mu\nu} + 4{}^*T_\mu{}^{\lambda\tau}T_{\lambda\tau\nu} = 0 \quad (12)$$

and

$$\nabla^{\rho*}R_{\rho\mu\lambda\sigma} + 2{}^*T_\mu{}^{\rho\tau}R_{\rho\tau\lambda\sigma} = 0. \quad (13)$$

In general, such dualizations must take up the covariantly constant world *tensors* $(-\mathfrak{g})^{1/2}\varepsilon_{\mu\nu\lambda\sigma}$ and $(-\mathfrak{g})^{-1/2}\varepsilon^{\mu\nu\lambda\sigma}$, with these ε -objects being the Levi-Civita world densities in \mathfrak{M} . Hence, according to Eq. (12), the typical contracted first-left dual pattern ${}^*R^\lambda{}_{\mu\lambda\nu}$ does not vanish, in contrast to the Riemann-Christoffel case.

3 Metric Spinors, Connecting Objects and Spin Affinities

One of the fundamental metric spinors for the γ -formalism is expressed by

$$(\gamma_{AB}) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \gamma = |\gamma| \exp(i\Phi). \quad (14)$$

By definition, it behaves as a spin tensor under gauge transformations, namely,

$$\gamma'_{AB} = \Lambda_A{}^C \Lambda_B{}^D \gamma_{CD} = \Delta_\Lambda \gamma_{AB}. \quad (15)$$

The polar components $|\gamma|$ and Φ are smooth real-valued world scalars, with $|\gamma| \neq 0$ throughout \mathfrak{M} . Their gauge behaviours will be described in a moment.

²In Ref. [18], this is unclearly posed.

For the inverse of (γ_{AB}) , one finds the expression

$$(\gamma^{AB}) = \begin{pmatrix} 0 & \gamma^{-1} \\ -\gamma^{-1} & 0 \end{pmatrix}, \quad (16)$$

together with the component relationships

$$\gamma_{AB} = \gamma \varepsilon_{AB}, \quad \gamma^{AB} = \gamma^{-1} \varepsilon^{AB}, \quad (17)$$

where

$$(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\varepsilon^{AB}). \quad (18)$$

The ε -spinors of (18) enter into the picture as world-invariant entities subject to the laws

$$\varepsilon'_{AB} = (\Delta_\Lambda)^{-1} \Lambda_A^C \Lambda_B^D \varepsilon_{CD} = \varepsilon_{AB} \quad (19)$$

and

$$\varepsilon'^{AB} = \Delta_\Lambda \varepsilon^{CD} \Lambda_C^{-1A} \Lambda_D^{-1B} = \varepsilon^{AB}, \quad (20)$$

whence ε_{AB} and ε^{AB} are invariant spin-tensor densities of weights -1 and $+1$, respectively. Thus, the entries (γ, γ^{-1}) appear as world-invariant spin-scalar densities of weight $(+1, -1)$ and, consequently, both of γ_{AB} and γ^{AB} bear world invariance as well. In accordance with these prescriptions, we have the coupled laws

$$|\gamma|' = \rho |\gamma| \quad (21)$$

and

$$\exp(i\Phi') = \rho^{-1} \Delta_\Lambda \exp(i\Phi), \quad (22)$$

together with

$$\partial'_\mu \Phi' = \partial_\mu \Phi + 2\partial_\mu \theta. \quad (23)$$

Any connecting objects for the γ -formalism satisfy anticommutation relations of the form

$$2\sigma_{AA'(\mu} \sigma_{\nu)}^{BA'} = \delta_A^B g_{\mu\nu}. \quad (24)$$

For the ε -formalism, we have

$$2\Sigma_{AA'(\mu} \Sigma_{\nu)}^{BA'} = \delta_A^B g_{\mu\nu}. \quad (25)$$

The entries of the set³

$$\mathbf{H} = \{S_{\mu AA'}, S_{AA'}^\mu, S_\mu^{AA'}, S^{\mu AA'}\}, \quad (26)$$

are components of Hermitian (2×2) -matrices that depend smoothly upon x^μ . Evidently, the Hermiticity of any element of the set (26) is lost when we let its spinor indices share out both stairs. Some of the most useful properties of the S -objects are expressed as

$$S_{\mu A'}^{(A} S_{\nu}^{B)A'} = S_{A'[\mu}^{(A} S_{\nu]}^{B)A'} = S_{A'[\mu}^A S_{\nu]}^{BA'}, \quad (27)$$

³The kernel letter S will henceforth stand for either σ or Σ .

whence we can likewise write

$$S_{AA'\mu} S_{\nu}^{AA'} = S_{AA'(\mu} S_{\nu)}^{AA'}. \quad (28)$$

The basic world-spin metric relations are thus given by

$$g_{\mu\nu} = S_{\mu}^{AA'} S_{\nu}^{BB'} M_{AB} M_{A'B'} \quad (29)$$

and

$$M_{AB} M_{A'B'} = S_{AA'}^{\mu} S_{BB'}^{\nu} g_{\mu\nu}, \quad (30)$$

whereas the spinor structure that represents the tensor $(-\mathfrak{g})^{1/2} \varepsilon_{\mu\nu\lambda\sigma}$ is written as⁴

$$e_{AA'BB'CC'DD'} = i(M_{AC} M_{BD} M_{A'D'} M_{B'C'} - \text{c.c.}), \quad (31)$$

with the kernel letter M denoting here as elsewhere either γ or ε . Every connecting object behaves as a vector as regards the action of the manifold mapping group of \mathfrak{M} . Those for the γ -formalism bear a gauge-tensor character while the Hermitian ones for the ε -formalism have to be regarded as invariant spin-tensor densities carrying the absolute weights ± 1 . For instance,

$$\sigma_{\mu A'}'^B = \Lambda_{A'}'^{B'} \sigma_{\mu B'}^C \Lambda_C^{-1B} = \exp(-2i\theta) \sigma_{\mu A'}^B \quad (32)$$

and

$$\Sigma_{AA'}'^{\mu} = \rho^{-1} \Lambda_A^B \Lambda_{A'}'^{B'} \Sigma_{BB'}^{\mu} = \Sigma_{AA'}^{\mu}. \quad (33)$$

We suppose that spin affinities in \mathfrak{M} bear an additivity property in both formalisms apart from the eventual implementation of any symmetry splittings. Typically, in either formalism, we have

$$\vartheta_{\mu AB} \doteq \tilde{\vartheta}_{\mu AB} + \vartheta_{\mu AB}^{(T)}. \quad (34)$$

In Eq. (34), $\tilde{\vartheta}_{\mu AB}$ is identified with the spin-affine connexion for the torsionless operator $\tilde{\nabla}_{\mu}$ whilst $\vartheta_{\mu AB}^{(T)}$ accounts for the torsionfulness of ∇_{μ} . Thus, for some spin vectors ζ^A and ξ_A , we have the corresponding patterns

$$\nabla_{\mu} \zeta^A = \tilde{\nabla}_{\mu} \zeta^A + \vartheta_{\mu B}^{(T)A} \zeta^B, \quad \nabla_{\mu} \xi_A = \tilde{\nabla}_{\mu} \xi_A - \vartheta_{\mu A}^{(T)B} \xi_B. \quad (35)$$

Towards making it feasible to ensure the self-consistency of the world-spin metric and affine structures carried by \mathfrak{M} , it is seemingly necessary to allow for the gauge-invariant constancy requirement

$$\nabla_{\lambda} S_{AA'}^{\mu} = 0. \quad (36)$$

In both formalisms, the spacetime metric compatibility condition then gets translated into

$$\nabla_{\mu} (M_{AB} M_{A'B'}) = 0. \quad (37)$$

⁴The symbol "c.c." has been taken throughout what follows to denote an overall complex conjugate piece.

The piece $\tilde{\vartheta}_{\mu AB}$ of the prescription (34) is required to carry only complex entries whence it contributes in either formalism 32 real independent components to $\vartheta_{\mu AB}$. It is apparently suggestive to think of the torsional piece of Eq. (34) as having the symmetry property $\vartheta_{\mu AB}^{(T)} = \vartheta_{\mu(AB)}^{(T)}$. This choice would at once supply the required 24 real independent components if it were actually taken into account. It has been used by some authors [4] for carrying out a rough spinor transcription of Einstein-Cartan's theory. Even though symmetry properties are gauge invariant, the symmetric choice for $\vartheta_{\mu AB}^{(T)}$ would nevertheless appear to be inadequate insofar as Eqs. (36) and (37) remain both unaltered when we add to each of $\tilde{\vartheta}_{\mu A}^B$ and $\vartheta_{\mu A}^{(T)B}$ purely imaginary world-covariant quantities⁵ of the type $\pm i\nu_\mu \delta_A^B$. As far as the situation at issue is concerned, the main point is that the implementation of a symmetric $\vartheta_{\mu AB}^{(T)}$ rules out all the possibilities of sorting out contracted torsional affinities to which one could eventually ascribe a physical meaning. Rather than implementing the symmetric torsional choice, we should make use of an asymmetric prescription that provides 20 real independent components along with four more coming from the purely imaginary trace

$$\vartheta_{\mu A}^{(T)A} = -2iA_\mu, \quad (38)$$

with A_μ thus being a world vector. A possible choice for $\vartheta_{\mu A}^{(T)B}$ is prescribed as

$$(\vartheta_{\mu A}^{(T)B}) = \begin{pmatrix} a_\mu & b_\mu \\ \beta_\mu & c_\mu \end{pmatrix}, \quad (39)$$

with the conditions

$$\text{Im } b_\mu = \text{Im } \beta_\mu = 0, \quad \vartheta_{\mu A}^{(T)A} = (a_\mu + c_\mu) \doteq -2iA_\mu. \quad (40)$$

Here, we disregard any torsional spin-affine prescription having $\text{Re}\vartheta_{\mu A}^{(T)A} \neq 0$, but every choice for $\vartheta_{\mu A}^{(T)B}$ is gauge invariant (see Eq. (49) below).

Of course, the patterns of $\vartheta_{\mu AB}^{(T)}$ and $\vartheta_{\mu A}^{(T)A}$ as stipulated by Eqs. (38)-(40) adequately carry $(20 + 4)$ real independent components in all. For $\tilde{\vartheta}_{\mu A}^A$, we similarly write the world-covariant prescription

$$\text{Im } \tilde{\vartheta}_{\mu A}^A = -2\Phi_\mu. \quad (41)$$

Hence, in both formalisms, we have the common piece

$$\text{Im } \vartheta_{\mu A}^A = -2(\Phi_\mu + A_\mu). \quad (42)$$

In the γ -formalism, Eq. (37) right away yields the four-real parameter relation

$$\text{Re } \tilde{\gamma}_{\mu A}^A = \partial_\mu \log |\gamma|. \quad (43)$$

⁵Such quantities should be the same in both formalisms. This was established in Ref. [18] for the case of the classical framework.

We observe that the right-hand side of (43) bears world covariance as $|\gamma|$ is a world-invariant real spin-scalar density. In the ε -formalism, there occurs no metric relation like (43) such that the respective piece $\text{Re } \tilde{\vartheta}_{\mu A}^A$ must be defined by hand such as in the classical framework (for further details, see Ref. [18]). In effect, we have the world-covariant ε -contribution

$$\text{Re } \tilde{\Gamma}_{\mu A}^A = \Upsilon_\mu. \quad (44)$$

It follows that $\tilde{\vartheta}_{\mu AB}$ and its trace contribute $(32 + 8)$ real independent components to the overall $\vartheta_{\mu AB}$. The pieces $(\tilde{\vartheta}_{\mu AB}, \tilde{\vartheta}_{\mu A}^A)$ and $(\vartheta_{\mu AB}^{(T)}, \vartheta_{\mu A}^{(T)A})$ are thus quantities that carry $(32, 8)$ and $(20, 4)$ real independent components in either formalism, and therefore recover the numbers of independent components of $\tilde{\Gamma}_{\mu\nu\lambda}$ and $T_{\mu\nu\lambda}$ appropriately.

To establish the gauge behaviours of any spin affinities for either formalism, we implement the covariant property

$$\nabla'_\mu \xi'_A = \Lambda_A{}^B \nabla_\mu \xi_B. \quad (45)$$

Writing out the expansions of (45) explicitly, after some differential manipulations, we end up with the law

$$\vartheta'_{\mu A}{}^B = \vartheta_{\mu A}{}^B + \frac{1}{2}(\partial_\mu \log \Delta_\Lambda) \delta_A{}^B, \quad (46)$$

whence, making a contraction over the indices A and B carried by (46), gives rise to

$$\vartheta'_{\mu A}{}^A = \vartheta_{\mu A}{}^A + \partial_\mu \log \Delta_\Lambda. \quad (47)$$

By this point, we call for the old procedure whereby any two-component spin-affine configurations should be built up so as to formally look like world ones.⁶ In both formalisms, the torsional piece $\vartheta_{\mu AB}^{(T)}$ should therefore behave covariantly under the action of the gauge group, which means that

$$\gamma_{\mu AB}^{(T)'} = \Delta_\Lambda \gamma_{\mu AB}^{(T)}, \quad \Gamma_{\mu AB}^{(T)'} = (\Delta_\Lambda)^{-1} \Lambda_A{}^C \Lambda_B{}^D \Gamma_{\mu CD}^{(T)} = \Gamma_{\mu AB}^{(T)}. \quad (48)$$

Thus, $\vartheta_{\mu A}^{(T)B}$ must bear gauge invariance in each formalism, that is to say,

$$\vartheta_{\mu A}^{(T)B} = \vartheta_{\mu A}^{(T)B}. \quad (49)$$

So, the behaviour of the torsionless piece $\tilde{\vartheta}_{\mu A}^B$ is such that it must absorb the inhomogeneous term lying on the right-hand side of Eq. (46), whence we should effectively combine (49) with

$$\tilde{\vartheta}'_{\mu A}{}^B = \tilde{\vartheta}_{\mu A}{}^B + \frac{1}{2}(\partial_\mu \log \Delta_\Lambda) \delta_A{}^B. \quad (50)$$

⁶This procedure really underlies the construction of the classical framework.

Calling upon Eqs. (15) and (19) then yields the laws

$$\tilde{\gamma}'_{\mu AB} = \Lambda_A{}^C \Lambda_B{}^D \tilde{\gamma}_{\mu CD} + \frac{1}{2}(\partial_\mu \Delta_\Lambda) \gamma_{AB} \quad (51)$$

and

$$\tilde{\Gamma}'_{\mu AB} = (\Delta_\Lambda)^{-1} \Lambda_A{}^C \Lambda_B{}^D \tilde{\Gamma}_{\mu CD} + \frac{1}{2}(\partial_\mu \log \Delta_\Lambda) \varepsilon_{AB}. \quad (52)$$

Obviously, the contracted versions of Eqs. (49) and (50) amount to the same thing as

$$A'_\mu = A_\mu, \quad \Phi'_\mu = \Phi_\mu - \partial_\mu \theta \quad (53)$$

and

$$\text{Re } \tilde{\vartheta}'_{\mu A}{}^A = \text{Re } \tilde{\vartheta}_{\mu A}{}^A + \partial_\mu \log \rho. \quad (54)$$

The construction of the affine devices for computing covariant derivatives of spin densities is based upon the requirement which amounts to looking upon the ε -metric spinors as covariantly constant objects in either formalism. This requirement is also implemented within the traditional Infeld-van der Waerden framework. We thus allow for a world-covariant quantity ϑ_μ defined by

$$\nabla_\mu \varepsilon_{AB} = 0 \Leftrightarrow \vartheta_\mu \equiv \tilde{\vartheta}_\mu + \vartheta_\mu^{(T)} = \vartheta_{\mu A}{}^A, \quad (55)$$

with $\tilde{\vartheta}_\mu \doteq \tilde{\vartheta}_{\mu A}{}^A$ and $\vartheta_\mu^{(T)} \doteq \vartheta_{\mu A}^{(T)A}$. Therefore, it also occurs in the formal configuration

$$\nabla_\mu \gamma_{AB} = \nabla_\mu (\gamma \varepsilon_{AB}) = \varepsilon_{AB} \nabla_\mu \gamma, \quad (56)$$

and likewise is made up by the expansion

$$\nabla_\mu \gamma = \tilde{\nabla}_\mu \gamma - \gamma \vartheta_\mu^{(T)}, \quad (57)$$

which constitutes the prototype in both formalisms for covariant derivatives of complex spin-scalar densities of weight +1. Clearly, the right-hand side of (57) stands for a covariant expansion for the independent component of γ_{AB} . For a complex spin-scalar density α of weight \mathfrak{w} in \mathfrak{M} , we then have

$$\nabla_\mu \alpha = \tilde{\nabla}_\mu \alpha - \mathfrak{w} \alpha \vartheta_\mu^{(T)}. \quad (58)$$

In case a density β carries the absolute weight $2\mathfrak{a}$, we will get the real expansion

$$\nabla_\mu \beta = \partial_\mu \beta - 2\mathfrak{a} \beta \text{Re } \vartheta_\mu = \tilde{\nabla}_\mu \beta, \quad (59)$$

whence, from Eqs. (21) and (43), we see that $|\gamma|$ is covariantly constant in the γ -formalism. Hence, all the Σ -objects bear covariant constancy in both formalisms.

In fact, the recovery in either formalism of covariant derivative patterns for arbitrary world tensors may only be achieved if the covariant constancy property (36) is accounted for. This condition allows us to deal with combined world-spin displacements in \mathfrak{M} . For instance,

$$\nabla_\mu u^\lambda = S_{AA'}^\lambda \nabla_\mu u^{AA'} \Leftrightarrow \nabla_\mu u^{AA'} = S_\lambda^{AA'} \nabla_\mu u^\lambda, \quad (60)$$

where u^λ amounts to a world vector. Some manipulations involving rearrangements of the index configurations of (60) then yield the general γ -formalism relationship

$$\Gamma_{\mu AA' BB'} + \sigma_{\lambda BB'} \partial_\mu \sigma_{AA'}^\lambda = \gamma_{\mu AB} \gamma_{A' B'} + \text{c.c.} \quad (61)$$

Hence, by recalling Eq. (6) and the spin-affine prescriptions given before, we get the correlation

$$4 \text{Re} \tilde{\gamma}_{\mu A}^A = \tilde{\Gamma}_\mu + T_\mu + \sigma_\lambda^{AA'} \partial_\mu \sigma_{AA'}^\lambda. \quad (62)$$

In the ε -formalism, $u^{AA'}$ is an Hermitian spin-tensor density of absolute weight +1, and one has the expansion

$$\nabla_\mu \Sigma_{AA'}^\lambda = \partial_\mu \Sigma_{AA'}^\lambda + \Gamma_{\mu\nu}^\lambda \Sigma_{AA'}^\nu - (\Gamma_{\mu A}^B \Sigma_{BA'}^\lambda + \text{c.c.}) + \Upsilon_\mu \Sigma_{AA'}^\lambda, \quad (63)$$

which can evidently be reset as

$$\nabla_\mu \Sigma_{AA'}^\lambda = \tilde{\nabla}_\mu \Sigma_{AA'}^\lambda + T_{\mu\nu}^\lambda \Sigma_{AA'}^\nu - (\Gamma_{\mu A}^{(T)B} \Sigma_{BA'}^\lambda + \text{c.c.}). \quad (64)$$

Therefore, the ε -formalism counterpart of (62) must be spelt out as

$$\tilde{\Gamma}_\mu + T_\mu + \Sigma_\lambda^{AA'} \partial_\mu \Sigma_{AA'}^\lambda = 0. \quad (65)$$

We end this Section by pointing out that the covariant constancy of the ε -metric spinors allows the implementation of the γ -formalism statement

$$\nabla_\mu \gamma_{AB} = (\gamma^{-1} \nabla_\mu \gamma) \gamma_{AB} = (\gamma^{-1} \tilde{\nabla}_\mu \gamma - \gamma_\mu^{(T)}) \gamma_{AB}, \quad (66)$$

which yields the expansion

$$\nabla_\mu \gamma_{AB} = (\partial_\mu \log \gamma - \gamma_\mu) \gamma_{AB}. \quad (67)$$

Since $\nabla_\mu \delta_A^B = 0$ invariantly, Eqs. (42) and (55) produce the covariant eigenvalue equations

$$\nabla_\mu \gamma_{AB} = i \alpha_\mu \gamma_{AB}, \quad \nabla_\mu \gamma^{AB} = -i \alpha_\mu \gamma^{AB}, \quad (68)$$

along with their complex conjugates and

$$\alpha_\mu = \partial_\mu \Phi + 2(\Phi_\mu + A_\mu). \quad (69)$$

It should be noticed that the behaviours specified by Eqs. (23) and (53) guarantee the gauge invariance of α_μ . Needless to say, the occurrence of purely imaginary eigenvalues in (68) reflects the applicability of the γ -formalism version of the condition (37).

4 Spin Curvatures

The mixed world-spin curvature object associated to either $\vartheta_{\mu AB}$ occurs in the differential configuration

$$D_{\mu\nu}\zeta^B = C_{\mu\nu A}{}^B \zeta^A, \quad (70)$$

where ζ^A is an arbitrary spin vector and $D_{\mu\nu}$ equals the operator given by (2). Hence, taking the second covariant derivative of ζ^A according to one of the expansions (35) and performing some calculational rearrangements, we get the pattern⁷

$$C_{\mu\nu A}{}^B = \tilde{C}_{\mu\nu A}{}^B + C_{\mu\nu A}^{(T) B} + \check{A}_{\mu\nu A}{}^B. \quad (71)$$

In Eq. (71), the contribution $\tilde{C}_{\mu\nu A}{}^B$ is identical to the one which occurs in the torsionless framework [16], namely,

$$\tilde{C}_{\mu\nu A}{}^B = 2\partial_{[\mu}\tilde{\vartheta}_{\nu]A}{}^B - (\tilde{\vartheta}_{\mu A}{}^C\tilde{\vartheta}_{\nu C}{}^B - \tilde{\vartheta}_{\nu A}{}^C\tilde{\vartheta}_{\mu C}{}^B), \quad (72)$$

and it just arises from

$$2\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}\zeta^B = \tilde{C}_{\mu\nu A}{}^B \zeta^A. \quad (73)$$

The quantity $C_{\mu\nu A}^{(T) B}$ takes account of the torsionfulness of $\vartheta_{\mu AB}$, with its defining expression being written as

$$C_{\mu\nu A}^{(T) B} = 2\partial_{[\mu}\vartheta_{\nu]A}^{(T) B} - (\vartheta_{\mu A}^{(T) C}\vartheta_{\nu C}^{(T) B} - \vartheta_{\nu A}^{(T) C}\vartheta_{\mu C}^{(T) B}). \quad (74)$$

In each formalism, the piece $\check{A}_{\mu\nu A}{}^B$ amounts to a spin-affine entanglement contribution which is typically given by

$$\check{A}_{\mu\nu A}{}^B = -[(\tilde{\vartheta}_{\mu A}{}^C\vartheta_{\nu C}^{(T) B} - \tilde{\vartheta}_{\nu A}{}^C\vartheta_{\mu C}^{(T) B}) + (\tilde{\vartheta}\vartheta^{(T)}\text{-piece})], \quad (75)$$

where the $\tilde{\vartheta}\vartheta^{(T)}$ -piece denotes the term that is obtained from the preceding one by interchanging the roles of the kernel letters $\tilde{\vartheta}$ and $\vartheta^{(T)}$.

Actually, a characteristic curvature splitting for the γ -formalism comes about in a straightforward way when we allow $D_{\mu\nu}$ to act freely upon any Hermitian σ -object. To see this, we allow for the γ -formalism version of (64) and work out the derivative $D_{\mu\nu}\sigma_{\lambda}^{AA'}$. After somewhat lengthy calculations, we thus obtain the intermediate-stage expansion

$$\begin{aligned} D_{\mu\nu}\sigma_{\lambda}^{AA'} &= 2\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}\sigma_{\lambda}^{AA'} - 2(\tilde{\nabla}_{[\mu}T_{\nu]\lambda}{}^{\rho} + T_{\lambda[\mu}{}^{\tau}T_{\nu]\tau}{}^{\rho})\sigma_{\rho}^{AA'} \\ &\quad + [2(\tilde{\nabla}_{[\mu}\gamma_{\nu]B}^{(T) A} - \gamma_{[\mu|B|}^{(T) C}\gamma_{\nu]C}^{(T) A})\sigma_{\lambda}^{BA'} + \text{c.c.}]. \end{aligned} \quad (76)$$

The torsionless second derivative of (76) possesses the same form as that of the traditional framework, that is to say,

$$2\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}\sigma_{\lambda}^{AA'} = -\tilde{R}_{\mu\nu\lambda}{}^{\rho}\sigma_{\rho}^{AA'} + (\tilde{C}_{\mu\nu B}{}^A\sigma_{\lambda}^{BA'} + \text{c.c.}), \quad (77)$$

⁷The object $C_{\mu\nu AB}$ carries 48 real independent components.

while the involved world-torsion piece amounts to

$$2(\tilde{\nabla}_{[\mu} T_{\nu]\lambda}{}^\rho + T_{\lambda[\mu}{}^\tau T_{\nu]\tau}{}^\rho) = R_{\mu\nu\lambda}^{(T)\rho} + \check{Z}_{\mu\nu\lambda}{}^\rho, \quad (78)$$

with $\check{Z}_{\mu\nu\lambda}{}^\rho$ being the world contribution (see Eq. (8))

$$\begin{aligned} \check{Z}_{\mu\nu\lambda}{}^\rho &= -[(\tilde{\Gamma}_{\mu\lambda}{}^\tau T_{\nu\tau}{}^\rho - \tilde{\Gamma}_{\nu\lambda}{}^\tau T_{\mu\tau}{}^\rho) + (\tilde{\Gamma}T\text{-piece})] \\ &= 2[(T_{[\mu|\tau|}{}^\rho \tilde{\Gamma}_{\nu]\lambda}{}^\tau + (\tilde{\Gamma}T\text{-piece})], \end{aligned} \quad (79)$$

and each of its $\tilde{\Gamma}T$ -pieces coming from an interchange similar to that of (75). In inserting $R_{\mu\nu\lambda}^{(T)\rho}$ into (78), it may be convenient to use the trivial equalities

$$T_{[\mu|\tau|}{}^\rho T_{\nu]\lambda}{}^\tau = -T_{[\mu|\lambda|}{}^\tau T_{\nu]\tau}{}^\rho = T_{\lambda[\mu}{}^\tau T_{\nu]\tau}{}^\rho. \quad (80)$$

Also, Eqs. (74) and (75) show us that the whole unprimed $\gamma^{(T)}$ -contribution of (76) reproduces the corresponding sum $C_{\mu\nu B}^{(T)A} + \check{A}_{\mu\nu B}{}^A$ whence, fitting pieces together, yields the expression

$$\begin{aligned} D_{\mu\nu}\sigma_\lambda^{AA'} &= -(\check{R}_{\mu\nu\lambda}{}^\rho + R_{\mu\nu\lambda}^{(T)\rho} + \check{Z}_{\mu\nu\lambda}{}^\rho)\sigma_\rho^{AA'} \\ &\quad + [(\check{C}_{\mu\nu B}{}^A + C_{\mu\nu B}^{(T)A} + \check{A}_{\mu\nu B}{}^A)\sigma_\lambda^{BA'} + \text{c.c.}], \end{aligned} \quad (81)$$

which suggests defining the formal expansion

$$D_{\mu\nu}\sigma_\lambda^{AA'} = -R_{\mu\nu\lambda}{}^\rho\sigma_\rho^{AA'} + (C_{\mu\nu B}{}^A\sigma_\lambda^{BA'} + \text{c.c.}), \quad (82)$$

in agreement with Eqs. (8) and (71). Then, transvecting (82) with $\sigma_{CA'}^\lambda$ leads to

$$2C_{\mu\nu A}{}^B + \delta_A{}^B C_{\mu\nu A'}{}^{A'} - \sigma_{AA'}^\lambda \sigma^{\rho BA'} R_{\mu\nu\lambda\rho} = 0, \quad (83)$$

which, in turn, brings about the property

$$\text{Re } C_{\mu\nu A}{}^A = 0, \quad (84)$$

provided that $R_{\mu\nu\lambda}{}^\lambda \equiv 0$. Consequently, since the contracted quadratic pieces of both (72) and (74) vanish identically together with $\check{A}_{\mu\nu B}{}^B$, we get the additivity relation

$$C_{\mu\nu A}{}^A = \tilde{C}_{\mu\nu A}{}^A + C_{\mu\nu A}^{(T)A}, \quad (85)$$

along with the purely imaginary twelve-parameter contribution

$$C_{\mu\nu A}{}^A = -2iF_{\mu\nu} \doteq -2i(\tilde{F}_{\mu\nu} + F_{\mu\nu}^{(T)}), \quad (86)$$

where

$$\tilde{F}_{\mu\nu} \doteq 2\partial_{[\mu}\Phi_{\nu]}, \quad F_{\mu\nu}^{(T)} \doteq 2\partial_{[\mu}A_{\nu]}, \quad (87)$$

with Eqs. (41)-(43) having been employed for expressing (87).

It is of interest to recast the pieces of Eq. (87) as

$$\tilde{F}_{\mu\nu} \doteq 2\tilde{\nabla}_{[\mu}\Phi_{\nu]}, \quad F_{\mu\nu}^{(T)} \doteq 2(\nabla_{[\mu}A_{\nu]} + T_{\mu\nu}{}^\lambda A_\lambda). \quad (88)$$

Hence, lowering the index B of Eq. (83), gives the splitting

$$C_{\mu\nu AB} = \frac{1}{2}\sigma_{AA'}^\lambda\sigma_B^{\rho A'}R_{\mu\nu\lambda\rho} - iF_{\mu\nu}\gamma_{AB}, \quad (89)$$

which recovers in the γ -formalism the number of real independent components of $C_{\mu\nu AB}$ as $36 + 12$. Because of the relation (27), the R -configuration of (89) bears symmetry in A and B such that

$$C_{\mu\nu(AB)} = \frac{1}{2}\sigma_{A'A}^\lambda\sigma_B^{\rho A'}R_{\mu\nu\lambda\rho}. \quad (90)$$

The derivation of the ε -formalism counterpart of Eq. (89) is carried out along the same lines as those yielding (82), but now we have to require

$$\nabla_{[\mu}(\Upsilon_{\nu]}\Sigma_\lambda^{AA'}) = 0 \Leftrightarrow \partial_{[\mu}\Upsilon_{\nu]} = T_{\mu\nu}{}^\lambda\Upsilon_\lambda. \quad (91)$$

In the classical framework, a similar requirement is also made which neatly fits in with the transformation law for the pertinent Υ_μ . Here, we naively ascribe a gauge-invariant character to (91) by choosing gauge matrices that possess constant modulus determinants, in which case we may write down the ε -expression

$$C_{\mu\nu AB} = \frac{1}{2}\Sigma_{AA'}^\lambda\Sigma_B^{\rho A'}R_{\mu\nu\lambda\rho} - iF_{\mu\nu}\varepsilon_{AB}. \quad (92)$$

In the γ -formalism, we thus have the tensor law

$$C'_{\mu\nu AB} = \Lambda_A^C\Lambda_B^D C_{\mu\nu CD} = \Delta_\Lambda C_{\mu\nu AB}, \quad (93)$$

whereas the object $C_{\mu\nu AB}$ for the ε -formalism must be taken as an invariant spin-tensor density of weight -1 , whence we also have

$$C'_{\mu\nu AB} = (\Delta_\Lambda)^{-1}\Lambda_A^C\Lambda_B^D C_{\mu\nu CD} = C_{\mu\nu AB}. \quad (94)$$

The curvature spinors for either formalism enter the bivector decomposition of the respective $C_{\mu\nu AB}$. We have, in effect,

$$S_{AA'}^\mu S_{BB'}^\nu C_{\mu\nu CD} = M_{A'B'}\varpi_{ABCD} + M_{AB}\varpi_{A'B'CD}, \quad (95)$$

along with the definitions

$$\varpi_{ABCD} = \varpi_{(AB)CD} \doteq \frac{1}{2}S_{A'A}^\mu S_B^{\nu A'} C_{\mu\nu CD} \quad (96)$$

and

$$\varpi_{A'B'CD} = \varpi_{(A'B')CD} \doteq \frac{1}{2}S_{AA'}^\mu S_{B'}^{\nu A} C_{\mu\nu CD}, \quad (97)$$

with the symmetries shown by (96) and (97) being once again ensured by Eq. (27). Each of the ϖ -curvatures of (95) obviously contributes 24 real independent components to $C_{\mu\nu CD}$. Owing to the gauge behaviours of the metric spinors and

C -objects, the curvature spinors for the γ -formalism are subject to the tensor laws

$$\varpi'_{ABCD} = \Lambda_A^L \Lambda_B^M \Lambda_C^R \Lambda_D^S \varpi_{LMRS} = (\Delta_\Lambda)^2 \varpi_{ABCD} \quad (98)$$

and

$$\varpi'_{A'B'CD} = \Lambda_{A'}^{L'} \Lambda_{B'}^{M'} \Lambda_C^R \Lambda_D^S \varpi_{L'M'RS} = \rho^2 \varpi_{A'B'CD}, \quad (99)$$

while the ones for the ε -formalism are invariant spin-tensor densities prescribed by

$$\varpi'_{ABCD} = (\Delta_\Lambda)^{-2} \Lambda_A^L \Lambda_B^M \Lambda_C^R \Lambda_D^S \varpi_{LMRS} = \varpi_{ABCD} \quad (100)$$

and

$$\varpi'_{A'B'CD} = \rho^{-2} \Lambda_{A'}^{L'} \Lambda_{B'}^{M'} \Lambda_C^R \Lambda_D^S \varpi_{L'M'RS} = \varpi_{A'B'CD}. \quad (101)$$

A glance at Eqs. (85), (86) and (92) tells us that the contracted curvature spinors $(\varpi_{ABC}^C, \varpi_{A'B'C}^C)$ for both formalisms constitute the bivector decomposition

$$-2iS_{AA'}^\mu S_{BB'}^\nu F_{\mu\nu} = M_{A'B'} \varpi_{ABC}^C + M_{AB} \varpi_{A'B'C}^C, \quad (102)$$

in addition to satisfying the property

$$\varpi_{ABC}^C = \tilde{\varpi}_{ABC}^C + \varpi_{ABC}^{(T)C}, \quad \varpi_{A'B'C}^C = \tilde{\varpi}_{A'B'C}^C + \varpi_{A'B'C}^{(T)C}. \quad (103)$$

Each of the pairs $(\tilde{\varpi}_{ABC}^C, \tilde{\varpi}_{A'B'C}^C)$ and $(\varpi_{ABC}^{(T)C}, \varpi_{A'B'C}^{(T)C})$ is now taken to contribute 6 real independent components to the overall $C_{\mu\nu A}^A$ of either formalism. Hence, making use of the torsional expression of (88) along with the prescriptions

$$T_{AA'BB'}^{CC'} A_{CC'} = M_{A'B'} \tau_{AB}^{CC'} A_{CC'} + \text{c.c.} \quad (104)$$

and

$$\tau_{AB}^{CC'} \doteq \frac{1}{2} T_{(AB)D'}^{D'CC'}, \quad (105)$$

we obtain the relationships

$$\varpi_{ABC}^{(T)C} = 2i(\nabla_{(A}^{C'} A_{B)C'} - 2\tau_{AB}^{CC'} A_{CC'}) \quad (106)$$

and⁸

$$\varpi_{A'B'C}^{(T)C} = 2i(\nabla_{(A'}^C A_{B')C} - 2\tau_{A'B'}^{CC'} A_{CC'}), \quad (107)$$

together with, say,

$$\tilde{\varpi}_{ABC}^C = 2i(\nabla_{(A}^{C'} \Phi_{B)C'} - 2\tau_{AB}^{CC'} \Phi_{CC'}). \quad (108)$$

Either of $\tau_{AB}^{CC'}$ and $\tau_{A'B'}^{CC'}$ recovers the number of independent components of $T_{\mu\nu}^\lambda$ as 4×6 . We must point out that the contracted curvature spinors for both formalisms obey the simultaneous conjugation relations

$$\varpi_{ABC}^C = -\varpi_{ABC'}^{C'}, \quad \varpi_{A'B'C}^C = -\varpi_{A'B'C'}^{C'}. \quad (109)$$

⁸We emphasize that, within our framework, $\tilde{\nabla}_\mu S_\lambda^{AA'} \neq 0$.

The Riemann curvature structure of \mathfrak{M} as defined by Eqs. (7)-(9) can be completely reinstated from the symmetric pair

$$\mathbf{R} = (\varpi_{AB(CD)}, \varpi_{A'B'(CD)}), \quad (110)$$

with each entry of which thus carrying 18 real independent components. The torsionless version of this statement was established in Ref. [18] out of utilizing some elementary metric formulae that may be formally applied to the case of (110) too. We can therefore recover as $18 + 6$ the number of degrees of freedom of each of the ϖ -spinors carried by Eq. (95). In both formalisms, we then have the gauge-covariant expression

$$R_{AA'BB'CC'DD'} = (M_{A'B'}M_{C'D'}\varpi_{AB(CD)} + M_{AB}M_{C'D'}\varpi_{A'B'(CD)}) + \text{c.c.}, \quad (111)$$

such that

$$\varpi_{AB(CD)} = \frac{1}{4}M^{A'B'}M^{C'D'}R_{AA'BB'CC'DD'} \quad (112)$$

and

$$\varpi_{A'B'(CD)} = \frac{1}{4}M^{AB}M^{C'D'}R_{AA'BB'CC'DD'}, \quad (113)$$

with the number of independent components of $R_{\mu\nu\lambda\sigma}$ accordingly appearing as $18 + 18$. It should be evident that the symmetries brought out by the configurations (96), (97) and (111) just correspond to the skew symmetry in the indices of the pairs $\mu\nu$ and $\lambda\sigma$ borne by $R_{\mu\nu\lambda\sigma}$. These are indeed the only symmetries carried by the curvature spinors (110). With the help of Eq. (31), we write the first-left dual of (111) as

$$^*R_{AA'BB'CC'DD'} = [(-i)(M_{A'B'}M_{C'D'}\varpi_{AB(CD)} - M_{AB}M_{C'D'}\varpi_{A'B'(CD)})] + \text{c.c.}, \quad (114)$$

whence the pair (110) may be directly obtained from the affine correlations of Section 3.

In either formalism, the number of degrees of freedom of $\varpi_{AB(CD)}$ becomes transparently visible when we put into effect the definitions

$$\varpi_{AB(CD)} \doteq X_{ABCD}, \quad \varpi_{A'B'(CD)} \doteq \Xi_{A'B'CD}, \quad (115)$$

along with the reduction device⁹

$$\begin{aligned} X_{ABCD} = & X_{(ABCD)} - \frac{1}{4}(M_{AB}X^L_{(LCD)} + M_{AC}X^L_{(LBD)} + M_{AD}X^L_{(LBC)}) \\ & - \frac{1}{3}(M_{BC}X^L_{A(LD)} + M_{BD}X^L_{A(LC)}) - \frac{1}{2}M_{CD}X_{AB}^L{}^L. \end{aligned} \quad (116)$$

Some calculations then yield the explicit expansion

$$X_{ABCD} = \Psi_{ABCD} - M_{(A|(C}\xi_{D)|B)} - \frac{1}{3}\kappa M_{A(C}M_{D)B}, \quad (117)$$

⁹In either of the old formalisms, the X-spinor carries $18 - 6 - 1$ degrees of freedom while the Ξ -spinor carries $10 - 1$ and bears Hermiticity. Within our framework, either Ξ -spinor can not be associated to any world tensor.

together with the individual pieces

$$\Psi_{ABCD} = X_{(ABCD)}, \quad \xi_{AB} = X^M_{(AB)M}, \quad \varkappa = X_{LM}{}^{LM}. \quad (118)$$

It is clear that the world-covariant character of $C_{\mu\nu AB}$ and the behaviours described by Eqs. (98)-(101), assure that \varkappa is a world-spin invariant in both formalisms. Moreover, since ${}^*R^\lambda_{\mu\lambda\nu} \neq 0$, we must regard \varkappa as a complex quantity. In essence, the factor 1/3 that occurs in the reduction of X_{ABCD} we have deduced, is due to the lack of symmetry of the piece X^M_{ABM} , in contraposition to the torsionless framework wherein the counterpart of the \varkappa -term of (117) carries a factor 2/3 because $\xi_{AB} \equiv 0$ thereabout. Hence, the pieces of (118) contribute (5, 3, 1) complex independent components to $\varpi_{AB(CD)}$, respectively. This component prescription was exhibited for the first time in Ref. [31]. For the Ricci tensor and scalar of ∇_μ , we thus get the expressions

$$R_{AA'BB'} = M_{AB}M_{A'B'} \operatorname{Re} \varkappa - [(M_{A'B'}\xi_{AB} + \Xi_{A'B'AB}) + \text{c.c.}] \quad (119)$$

and

$$R = 4 \operatorname{Re} \varkappa, \quad (120)$$

with Eq. (119) recovering the number of degrees of freedom of $R_{\mu\nu}$ as 1 + 6 + 9. Likewise, for the contracted first-left dual ${}^*R^\lambda_{\mu\lambda\nu}$, we have

$${}^*R^{CC'}_{AA'CC'BB'} = [i(M_{A'B'}\xi_{AB} - \frac{1}{2}M_{AB}M_{A'B'}\varkappa - \Xi_{A'B'AB})] + \text{c.c.}, \quad (121)$$

whence

$${}^*R_{\mu\nu}{}^{\mu\nu} = 4 \operatorname{Im} \varkappa. \quad (122)$$

We shall now proceed to deriving the spinor version of Eqs. (12) and (13). Let us write the dual spinor torsion

$${}^*T_{AA'BB'CC'} = i(M_{AB}\tau_{A'B'CC'} - \text{c.c.}), \quad (123)$$

with its τ -pieces being defined explicitly by Eq. (105). In the γ -formalism, the world derivative $\nabla^\lambda {}^*T_{\lambda\mu\nu}$ then corresponds to

$$\nabla^{AA'} {}^*T_{AA'BB'CC'} = i[(\nabla_B^{A'}\tau_{A'B'CC'} + i\alpha_B^{A'}\tau_{A'B'CC'}) - \text{c.c.}], \quad (124)$$

where Eq. (68) has been utilized. The ε -formalism version of (124) may be obtained by simply dropping the α -term from it, namely,

$$\nabla^{AA'} {}^*T_{AA'BB'CC'} = i(\nabla_B^{A'}\tau_{A'B'CC'} - \text{c.c.}). \quad (125)$$

We next write the *TT -kernel of Eq. (12) as

$$\begin{aligned} & {}^*T_{BB'}{}^{DD'MM'}T_{DD'MM'CC'} \\ &= i[(\tau_B{}^{DM}{}_{B'}\tau_{DMCC'} - \text{c.c.}) - (\tau_{BD}{}^{DD'}\tau_{B'D'CC'} - \text{c.c.})]. \end{aligned} \quad (126)$$

In both formalisms, the combination of Eqs. (124)-(126) with the expression (121) enables one to express the pair (110) in terms of spin-torsion constituents and their derivatives, as had been alluded to in Ref. [31]. For the contracted derivative involved in Eq. (13), we have the reduced contribution

$$\begin{aligned} & M^{C'D'} \nabla^{AA'} * R_{AA'BB'CC'DD'} \\ &= (-2i)(\nabla_{B'}^A X_{ABCD} - 2i\alpha_{B'}^A X_{ABCD} - \nabla_B^{A'} \Xi_{A'B'CD}), \end{aligned} \quad (127)$$

together with the complex conjugate of it. Whence, the Bianchi identity can be recovered by combining (127) with

$$\begin{aligned} & M^{C'D'} * T_{BB'}^{LL'MM'} R_{LL'MM'CC'DD'} \\ &= 2i[(\tau_{B'L'}^{LL'} X_{LBCD} - \tau_{B'}^{(L'M')} \Xi_{L'M'CD}) \\ & \quad + (\tau_B^{(LM)} X_{LMCD} - \tau_{BL}^{LL'} \Xi_{L'B'CD})]. \end{aligned} \quad (128)$$

5 Concluding Remarks and Outlook

According to the classical world geometry, any torsion tensors may be suppressed from general affine prescriptions while symmetric affinities may not. This is because the inhomogeneous parts coming from partial derivatives are strictly cancelled by those carried by symmetric connexions. Thus, the geometric adequacy of the equality (8) stems from the world-tensor character of Eq. (78). It is worth remarking that any non-contracted curvatures may not enjoy the additivity property. This feature has necessarily to be carried over to $R_{\mu\nu}$ and R as the configuration (79) appropriately yields both $\tilde{Z}_{\lambda\mu}^{\lambda}{}_{\nu} \neq 0$ and $\tilde{Z}_{\mu\nu}{}^{\mu\nu} \neq 0$. Therefore, we can roughly say that the property (85) ceases holding for the world case.

Since any two-component spinor approach to curved spacetimes should formally resemble the traditional world ones, we can infer that any torsional affinities like those we have defined in Section 3 must always be accompanied by suitable torsionless $\gamma\varepsilon$ -affine contributions. It is upon this fact that the genuineness of the gauge laws (48) and (49) rests. Hence, as the contracted affinity $\vartheta_{\mu A}^{(T)A}$ for either formalism has been chosen such that $\text{Re } \vartheta_{\mu}^{(T)} = 0$, we can conclude that the limiting procedure which could be implemented hereabout takes world and spin torsion contributions to vanish independently of one another.

We realize that the Bianchi identity as exhibited by Eqs. (127) and (128) could not only bring out the most characteristic form of two-component spinor couplings between curvatures and torsion, but could also afford the field equations which control the propagation of gravitons in torsional environments. With regard to this latter situation, the torsionful extension of the differential calculational techniques used in Refs. [22, 23] for deriving the wave equations of the torsionless framework, would be of the utmost significance. Such techniques may of course be additionally utilized to describe the interaction in the presence of torsion between the cosmic microwave background and Dirac particles.

In our view, the inner structure of the torsional spinor formalisms just constructed could provide locally a realistic description of the cosmic dark energy through a gauge-invariant potential like that defined by Eq. (38), while at the same time assigning geometrically a clear physical meaning to the right-hand side of Einstein-Cartan's field equations. In spacetimes having $R = 0$, the expressions (119) and (120) for either formalism particularly yield a purely imaginary \varkappa -quantity together with a traceless energy-momentum tensor $E_{\mu\nu}$ and the statement

$$(M_{A'B'}\xi_{AB} + \Xi_{A'B'AB}) + \text{c.c.} = \kappa E_{AA'BB'}.$$

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